SOME COMMENTS ON R.L.SALGANIK'S PAPER "ESTIMATING THE ERROR COMMITTED IN PASSING FROM THE EXACT EQUATIONS OF THE THEORY OF ELASTICITY TO THE EQUATIONS OF THE PLANE STATE OF STRESS"

(NEKOTORYE ZAMECHANIIA K RABOTE R.L. SALGANIKA "OB OTSENKE OSHIBKI, SOVERSHAEMOI PRI PEREKHODE OT TOCHNYKH URAVNENII TEORII UPRUGOSTI K URAVNENIIAM PLOSKOGO NAPRIAZHENNOGO SOSTOIANIIA" (PMM VOL.28, № 4, 1964))

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1. In the above paper an estimate is given of the asymptotic error of conversion from three-dimensional problems of the theory of elasticity to two-dimensional problems of the plane state of stress. The author claims his estimate cannot be improved. On the basis of these results an attempt is made to explain why, in certain cases [1], the subsequent terms of the asymptotic expansion appear more significant than the early terms. However, the estimate itself and especially the author's claim that it cannot be improved, may cause misunderstanding in this important question. It is therefore expedient to examine certain aspects of the paper [2].

For simplicity we will examine a finite plate, bounded by plane faces Γ and the curvilinear cylinder Γ^* with a closed directrix γ. Paper [2] assumes that Γ are free of stresses and that on Γ^* the displacement vector is given, for which (*)

$$u_\alpha^*\big|_\Gamma = \varphi_\alpha(\xi_1, \xi_2) \qquad (\alpha = 1, 2) \tag{1}$$

$$u_3^*\big|_\Gamma = 0 \tag{2}$$

An approximate solution of the three-dimensional problem is taken in the form

$$u_\alpha = u_{\alpha 0} + \frac{\lambda}{3(\lambda + 2\mu)} \frac{\partial \theta_0}{\partial \xi_\alpha} P_2(\xi_3), \qquad u_3 = -\frac{\lambda}{\lambda + 2\mu}\theta_0 P_1(\xi_3) \qquad \left(\theta_0 = \frac{\partial u_{\alpha 0}}{\partial \xi_\alpha}\right) \tag{3}$$

where $u_{\alpha 0}$ is the solution of the plane state of stress problem corresponding to condition (1). Estimating the difference $\delta u_i = u_i - u_i^*$, the author assumes known solutions for the problems for u_i and u_i^* and extends these vectors smoothly in a certain vicinity of Γ^* so that δu_i become zero at a distance

*) All notations used below are from paper [1] unless indicated.

of the order of 1 from Γ^*, and so that the faces of Γ remain free of stress. It is easy to see that

$$\delta u_i(P) = \int_D G_{ik}(P, Q)\, f_k(Q)\, dQ \tag{4}$$

Here G_{ik} is Green's tensor for the membrane and f_k is a linear combination of θ_0 and its derivatives up to the third order inclusive. Applying Buniakowski's inequality to (4) and assuming that $G_{ik} = O(\ln \rho)$, Salganik presents the following estimation:

$$\delta u_i = O(M \ln \Lambda\ \Lambda), \qquad M = \max\left\{\theta_0,\ \frac{\partial\theta_0}{\partial\xi_\alpha},\ \frac{\partial^2\theta_0}{\partial\xi_\alpha\partial\xi_\beta},\ \frac{\partial^3\theta_0}{\partial\xi_\alpha\partial\xi_\beta\partial\xi_\gamma}\right\} \tag{5}$$

2. First, we note that the possibility of smooth extension of the vectors u_i, $u_i{}^*$ (which represent the solution of the corresponding boundary value problem) beyond the boundaries of region Ω of the plate under consideration, is not obvious. See [3 and 4] for the possibility of such extensions. However, in the present case, we must be aware that the matter is complicated by the requirement for an absence of stresses on Γ for the extended vector.

Nevertheless, we admit, following Salganik, that such an extension is possible and we return to Equation (5). A more careful analysis shows that upon calculation to the order of Λ of the right-hand side of Formula (5) a factor of Λ is omitted. Indeed, using Buniakowski's inequality from (4) we obtain

$$|\delta u_i| \leqslant \left\{\int_D \sum_k |G_{ik}|^2\, dQ\right\}^{1/2} \left\{\int_D \sum_k |f_k|^2\, dQ\right\}^{1/2} \tag{6}$$

Assuming further that $G_{ik} = O(\ln \rho)$ we have

$$\left\{\int_D \sum_k |G_{ik}|^2\, dQ\right\}^{1/2} \sim \left(\int_D \ln^2 \rho\, dQ\right)^{1/2} \sim \left(2\pi \int_0^\Lambda \ln^2\rho\ \rho\, d\rho\right)^{1/2} = O(\Lambda \ln \Lambda) \tag{7}$$

Further, since all f_k are bounded in the region of integration by one and the same constant M, we have the obvious relationship

$$\left\{\int_D \sum_k |f_k|^2\, dQ\right\}^{1/2} = O(M \cdot \Lambda) \tag{8}$$

From (7) and (8) it follows that

$$\delta u_i = O(M\Lambda^2 \ln \Lambda) \tag{9}$$

instead of (5). The statement concerning the impossibility of improving the estimates is also erroneous. Indeed, in the demonstration, Salganik proceeds from the fact that Buniakowski's inequality is transformed into an equality if the functions entering into it differ by a numerical factor. Thus, under the conditions in question here, the estimate (9) may be obtained exactly if $f_k = \mathrm{const}\ G_{ik}$. But the latter is impossible for at least two reasons. Firstly, in structure, f_k vanish at a distance of the order of 1 from Γ^* and G_{ik} does not possess such properties. Secondly, G_{ik} has inside the region Ω certain singularities; whereas f_k in the region of the plate are holomorphic (since $u_{0\alpha}$ are solutions of plane elasticity problems) and are easily extended beyond the plate. Thus, the assertion about the finality of estimates of type (9) is not valid.

We call attention to the fact that estimates of type (9) (if one still considers the author's claim concerning the impossibility of improving them, may produce an impression that the error of the hypothesis about the plane state of stress in the determination of displacement, increases infinitely when $h \to 0$. In fact, there is no basis for such an affirmation since the estimate (9) is overstated. Detailed investigation [5 and 6] of cases of bending of the plate shows that the state of stress with the Γ free of stresses may be represented symbolocally in the form

$$H = H_1 + H_2, \qquad H_1 = \sum_{k=p}^{\infty} H_{1k}(\xi_1, \xi_2, \xi_3)\Lambda^{-k}$$

$$H_2 = \sum_{m=1}^{\infty} e^{-\Lambda\gamma_m n} \sum_{k=p}^{\infty} b_{mk}(\xi_1, \xi_2)\, a_m(\xi_3)\Lambda^{-k} \qquad (\sin 2\gamma_m + 2\gamma_m)\frac{\sin\gamma_m}{\gamma_m^2} = 0 \tag{10}$$

In (10) the terms H_{1k} and b_{sk} are continuous functions of the coordinates, n is the distance along the inner normal. Any term of the series expansion (10) may be written if γ and the external loading are infinitely differentiable. Such a case appears in [1]. Analysis of type (10) may also be obtained for displacement under conditions [2].

For this $p = 0$. When γ and $u_\alpha(\xi_1, \xi_2)$ are differentiable only a finite number of times, the analysis (10) may be discontinued with a proper approximation.

From the above mentioned it follows that for investigation of errors of plane state of stress it is necessary to consider the differential properties of γ and of given displacenents u_i on Γ^*; moreover, the estimation of errors will differ essentially if closed areas not having common points with Γ^* are considered or if regions extending on Γ^* are considered. These circumstances are ignored in paper [2]. One can only guess that γ and $u_\alpha|_{\Gamma^*}$ must be sufficiently smooth here. Indeed, $u_{\alpha 0}$, being the solutions of a system of equations of plane elastic problems, possess after Salganik continuous derivatives of the fourth order in a closed plane area bounded by γ. This is possible if γ and $u_\alpha|_{\Gamma^*}$ are sufficiently smooth. Therefore, according to recent results on this question [7] the magnitude of γ should be within Ω_4, and u_α should belong to class $H(4, A, \sigma)$. For the determination of Ω_4 and $H(4, A, \sigma)$ see, e.g. paper [8]. Using the method developed in (5), we can establish that in this case $\delta u_i = O(1)$ in the closed region, occupied by the plate. If it is assumed that γ and $u_\alpha|_\Gamma$ have stronger differential properties, then we can write a larger number of terms in the series. We note in conclusion that the estimation of the remaining terms in this case may, apparently, include terms of the form $\Lambda^{-t}\ln\Lambda$; $t > 0$. However, such an estimation may be obtained in connection with sufficiently fine characteristics of differential properties of γ and $u_\alpha|_{\Gamma^*}$. These characteristics should, for example, include the property of functions, which belong to the same class of Hölder's condition.

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CORRECTION TO THE PAPER "A THEOREM OF DYNAMICS"

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In my note "A theorem of dynamics" there is a fundamental error in Formula (9), which makes the basic result of the paper, namely Formula (12), erroneous in the general case.

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